When doing volume of revolution problems,
how do you decide which way to cut the shape, AND
which method (disk / washer / shell) to use to set up the integral ?

[1] How do you decide which direction to cut the region into strips ?

NOTE: a boundary is an edge of the base region defined by a single formula

Overriding criteria (these criteria require you to cut in a specific direction)

If there is a boundary which you cannot solve as $x = g(y)$ or $y = k$, you MUST cut vertically (dx)

If there is a boundary which you cannot solve as $y = f(x)$ or $x = c$, you MUST cut horizontally (dy)

Preferred criteria (if the overriding criteria do not require you to cut in a specific direction):

If possible, cut so that you never cross the perimeter more than twice with any cut
to avoid getting washers within washers, or shells within shells

If possible, cut so that you never cross a single boundary more than once with any cut
to avoid having to convert a function in one variable into 2 functions in the other variable

If possible, cut so that you never cross different boundaries with different cuts
to avoid having to write multiple integrals

If possible, cut so that the integrand is simpler
to avoid having to find the anti-derivatives of complex functions

NOTE: the decision on which direction to cut depends on the region being revolved, NOT the axis of revolution

[2] How do you find the method of integration after you've decided which direction to cut ?

If the cut is parallel to the axis of revolution

use the shell method

If the cut is perpendicular to the axis of revolution

if the axis of revolution is a boundary of the region

use the disk method

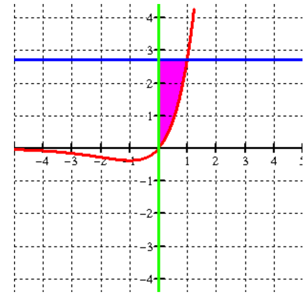
if the axis of revolution is not a boundary of the region

use the washer method

Examples of how to decide which direction to cut

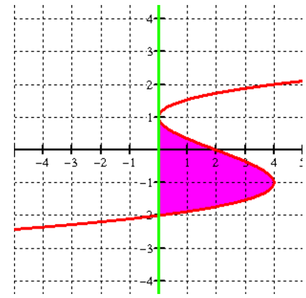
Region bounded by $y = xe^x$, $x = 0$ and $y = e$ revolved around any axis

Since we can't solve the boundary $y = xe^x$ as $x = g(y)$,
we can't write a dy integral.
So, we MUST cut vertically to create a dx integral.



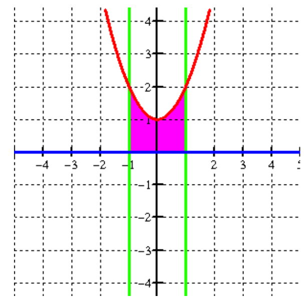
Region bounded by $x = y^3 - 3y + 2$ and $x = 0$ revolved around any axis

Since we can't solve the boundary $x = y^3 - 3y + 2$ as $y = f(x)$,
we can't write a dx integral.
So, we MUST cut horizontally to create a dy integral.



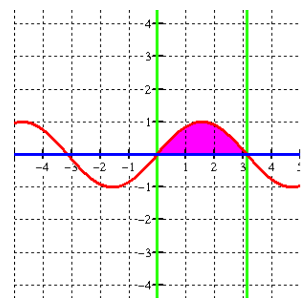
Region bounded by $y = 1 + x^2$, $y = 0$, $x = -1$ and $x = 1$ revolved around any axis

Since we can solve the boundary $y = 1 + x^2$ as $x = \pm\sqrt{y-1}$,
we write either a dx or dy volume integral,
so we can cut vertically or horizontally.
However, if we cut horizontally, the cut will cross the perimeter 4 times when $1 \le y \le 2$,
creating 2 washers (one within the other) or 2 shells.
This doesn't happen when we cut vertically, so we prefer to cut vertically.



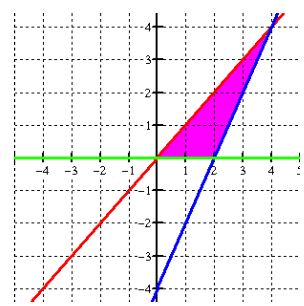
Region bounded by $y = \sin x$, $y = 0$, $x = 0$ and $x = \pi$ revolved around any axis

If we cut horizontally, the cut will cross the boundary $y = \sin x$ twice,
so we need to write $y = \sin x$ as 2 different functions $x = g_1(y)$ and $x = g_2(y)$.
This is possible, but not necessarily easy, to do.
This doesn't happen when we cut vertically, so we prefer to cut vertically.



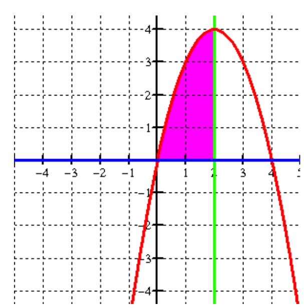
Region bounded by $y = x$, $y = 2x - 4$ and $y = 0$ revolved around any axis

If we cut vertically, the cut will cross the boundaries $y = x$ and $y = 0$ when $0 \le x \le 2$,
but the cut will cross the boundaries $y = x$ and $y = 2x - 4$ when $2 \le x \le 4$,
so we need to write 2 different integrals.
When we cut horizontally, the cut will only cross the boundaries $y = x$ and $y = 2x - 4$,
so we only need to write 1 integral.
So, we prefer to cut horizontally.



Region bounded by $y = 4 - (x-2)^2 = 4x - x^2$, $y = 0$ and $x = 2$ revolved around any axis

Since we can solve the boundary $y = 4 - (x-2)^2$ as $x = 2 - \sqrt{4 - y}$,
we write either a dx or dy volume integral,
so we can cut vertically or horizontally.
But, $x = 2 - \sqrt{4 - y}$ may be a more complex function to work with than $y = 4x - x^2$.
So, we prefer to cut vertically to create a dx integral.



Volume for revolved solids (using discs / washers)

Draw the region to be revolved, the axis of revolution, and the reflection of the region over the axis

Cut the original region perpendicular to the axis of revolution
and connect the ends of the cut to their reflections using circles

Find the area of the cross section

πr^2 if the axis of revolution is a boundary of the original region (disc method)
where r is the distance from the axis to the opposite boundary

$\pi(R^2 - r^2)$ if the axis of revolution is not a boundary of the original region (washer method)
where R is the distance from the axis to the farther endpoint of the cut
and r is the distance from the axis to the closer endpoint of the cut

Find the least (a) and greatest (b) values of the variable (look at the original region)

Integrate the area formula from a to b

Volume for revolved solids (using shells)

Draw the region to be revolved, the axis of revolution, and the reflection of the region over the axis

Cut the original region parallel to the axis of revolution and connect the ends of the cut to their reflections using circles

Find the radius (r) (from the axis of revolution to the cut) and the height (h) (the length of the cut) of the shell

Find the least (a) and greatest (b) values of the variable (look at the original region)

Integrate $2\pi rh$ from a to b

Volume for non-revolved solids with fixed shape cross sections

Draw the base region

Cut the region in the direction specified

Draw the cross section

Find a formula for the area of the cross section in terms of the length of the cut (s)

Find a formula for the length of the cut (s) in terms of x (if the cut was perpendicular to the x-axis)
or y (if the cut was perpendicular to the y-axis)

Substitute for s in the area formula to get a new area formula in terms of x or y

Find the least (a) and greatest (b) values of the remaining variable (look at the base region)

Integrate the area formula from a to b